Schwarzschild metric
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In Einstein's theory of general relativity, the Schwarzschild solution (or the Schwarzschild vacuum) describes the gravitational field outside a spherical, uncharged, non-rotating mass such as a (non-rotating) star, planet, or black hole. It is also a good approximation to the gravitational field of a slowly rotating body like the Earth or Sun. The cosmological constant is assumed to equal zero.

According to Birkhoff's theorem, the Schwarzschild solution is the most general spherically symmetric, vacuum solution of the Einstein field equations. A Schwarzschild black hole or static black hole is a black hole that has no charge or angular momentum. A Schwarzschild black hole has a Schwarzschild metric, and cannot be distinguished from any other Schwarzschild black hole except by its mass.

The Schwarzschild black hole is characterized by a surrounding spherical surface, called the event horizon, which is situated at the Schwarzschild radius, often called the radius of a black hole. Any non-rotating and non-charged mass that is smaller than its Schwarzschild radius forms a black hole. The solution of the Einstein field equations is valid for any mass \( M \), so in principle (according to general relativity theory) a Schwarzschild black hole of any mass could exist if conditions became sufficiently favorable to allow for its formation.

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### History

The Schwarzschild solution is named in honor of Karl Schwarzschild, who found the exact solution in 1915, only about a month after the publication of Einstein's theory of general relativity.\(^{[1]}\) It was the first exact solution of the Einstein field equations other than the trivial flat space solution. Schwarzschild had little time to think about his solution. He died shortly after his paper was published, as a result of a disease he
contracted while serving in the German army during World War I.

The original Schwarzschild solution\[^2\] used a different radial coordinate system than present formulations of the Schwarzschild metric. Schwarzschild's coordinate system has its origin at what is now recognized as the event horizon, with his radial coordinate \( r \) related to the conventional radial coordinate \( R \) (in Schwarzschild's notation) by \( R = \left( r^3 + \alpha^3 \right)^{1/3} \). The value \( \alpha \), a constant of integration in Schwarzschild's paper, corresponds to what is now called the Schwarzschild radius \( r_s \). As a result of this choice of coordinate system, the original solution did not reach all the way to the center of the black hole where the gravitational singularity lies, stopping instead at the event horizon.\[^3\] The current version of the solution is due to David Hilbert.\[^4\]

**The Schwarzschild metric**

*Main article: Deriving the Schwarzschild solution*

In Schwarzschild coordinates, the **Schwarzschild metric** has the form:

\[
c^2 d\tau^2 = \left( 1 - \frac{r_s}{r} \right) c^2 dt^2 - \left( 1 - \frac{r_s}{r} \right)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right)
\]

where:

- \( \tau \) is the proper time (time measured by a clock moving with the particle) in seconds,
- \( c \) is the speed of light in meters per second,
- \( t \) is the time coordinate (measured by a stationary clock at infinity) in seconds,
- \( r \) is the radial coordinate (circumference of a circle centered on the star divided by \( 2\pi \)) in meters,
- \( \theta \) is the colatitude (angle from North) in radians,
- \( \varphi \) is the longitude in radians, and
- \( r_s \) is the Schwarzschild radius (in meters) of the massive body, which is related to its mass \( M \) by
  \[
r_s = \frac{2GM}{c^2}, \text{ where } G \text{ is the gravitational constant.}\[^5\]
\]

The analogue of this solution in classical Newtonian theory of gravity corresponds to the gravitational field around a point particle.\[^6\]

In practice, the ratio \( r_s/r \) is almost always extremely small. For example, the Schwarzschild radius \( r_s \) of the Earth is roughly 8.9 millimeters (0.35 in), whereas a satellite in a geosynchronous orbit has a radius \( r \) that is roughly four billion times larger, at 42,164 kilometers (26,199 mi). Even at the surface of the Earth, the corrections to Newtonian gravity are only one part in a billion. The ratio only becomes large close to black holes and other ultra-dense objects such as neutron stars.

The Schwarzschild metric is a solution of Einstein's field equations in empty space, meaning that it is valid only outside the gravitating body. That is, for a spherical body of radius \( R \) the solution is valid for \( r > R \). To describe the gravitational field both inside and outside the gravitating body the Schwarzschild solution must be matched with some suitable interior solution at \( r = R \).

When considering an object falling into a black hole, it is better to use a different coordinate system such as Kruskal–Szekeres coordinates.
Alternative (isotropic) formulations of the Schwarzschild metric

The original form of the Schwarzschild metric involves anisotropic coordinates, in terms of which the velocity of light is not the same for the radial and transverse directions (pointed out by A S Eddington).[7] Eddington gave alternative formulations of the Schwarzschild metric in terms of isotropic coordinates (provided \( r \geq 2GM/c^2 \).[8])

In isotropic spherical coordinates, one uses a different radial coordinate, \( r_1 \), instead of \( r \). They are related by

\[
    r = r_1 \left(1 + \frac{GM}{2c^2 r_1}\right)^2.
\]

Using \( r_1 \), the metric is

\[
    c^2 d\tau^2 = \left(1 - \frac{GM}{2c^2 r_1}\right)^2 \left(1 + \frac{GM}{2c^2 r_1}\right)^4 \left(\frac{1}{1 + \frac{GM}{2c^2 r_1}}\right)^2 c^2 dt^2 - \left(1 + \frac{GM}{2c^2 r_1}\right)^4 \left(dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2 \theta \, d\varphi^2\right).
\]

Then for isotropic rectangular coordinates \( x, y, z \), where

\[
    x = r_1 \sin \theta \cos \phi, \quad y = r_1 \sin \theta \sin \phi, \quad z = r_1 \cos \theta,
\]

and

\[
    r_1 = \sqrt{x^2 + y^2 + z^2},
\]

the metric then becomes

\[
    c^2 d\tau^2 = \left(1 - \frac{GM}{2c^2 r_1}\right)^2 \left(1 + \frac{GM}{2c^2 r_1}\right)^4 \left(\frac{1}{1 + \frac{GM}{2c^2 r_1}}\right)^2 c^2 dt^2 - \left(1 + \frac{GM}{2c^2 r_1}\right)^4 (dx^2 + dy^2 + dz^2).
\]

In the terms of these coordinates, the velocity of light at any point is the same in all directions, but it varies with radial distance \( r_1 \) (from the point mass at the origin of coordinates), where it has the value

\[
    \frac{1 - \frac{GM}{2c^2 r_1}}{\left(1 + \frac{GM}{2c^2 r_1}\right)^3} c. \quad [7]
\]

Singularities and black holes

The Schwarzschild solution appears to have singularities at \( r = 0 \) and \( r = r_s \); some of the metric components blow up at these radii. Since the Schwarzschild metric is only expected to be valid for radii larger than the radius \( R \) of the gravitating body, there is no problem as long as \( R > r_s \). For ordinary stars...
and planets this is always the case. For example, the radius of the Sun is approximately 700,000 km, while its Schwarzschild radius is only 3 km.

One might naturally wonder what happens when the radius $R$ becomes less than or equal to the Schwarzschild radius $r_s$. It turns out that the Schwarzschild solution still makes sense in this case, although it has some rather odd properties. The apparent singularity at $r = r_s$ is an illusion; it is an instance of what is called a *coordinate singularity*. As the name implies, the singularity arises from a bad choice of coordinates or coordinate conditions. By choosing another set of suitable coordinates one can show that the metric is well-defined at the Schwarzschild radius. See, for example, Lemaître coordinates, Eddington-Finkelstein coordinates, Kruskal-Szekeres coordinates, Novikov coordinates, or Gullstrand–Painlevé coordinates.

The case $r = 0$ is different, however. If one asks that the solution be valid for all $r$ one runs into a true physical singularity, or *gravitational singularity*, at the origin. To see that this is a true singularity one must look at quantities that are independent of the choice of coordinates. One such important quantity is the Kretschmann invariant, which is given by

$$K^{\alpha \beta \gamma \delta} = \frac{12r_s^2}{r^6} = \frac{48G^2M^2}{c^4r^6}.$$  

At $r = 0$ the curvature blows up (becomes infinite) indicating the presence of a singularity. At this point the metric, and space-time itself, is no longer well-defined. For a long time it was thought that such a solution was non-physical. However, a greater understanding of general relativity led to the realization that such singularities were a generic feature of the theory and not just an exotic special case. Such solutions are now believed to exist and are termed *black holes*.

The Schwarzschild solution, taken to be valid for all $r > 0$, is called a **Schwarzschild black hole**. It is a perfectly valid solution of the Einstein field equations, although it has some rather bizarre properties. For $r < r_s$ the Schwarzschild radial coordinate $r$ becomes timelike and the time coordinate $t$ becomes spacelike. A curve at constant $r$ is no longer a possible worldline of a particle or observer, not even if a force is exerted to try to keep it there; this occurs because spacetime has been curved so much that the direction of cause and effect (the particle's future light cone) points into the singularity[citation needed]. The surface $r = r_s$ demarcates what is called the *event horizon* of the black hole. It represents the point past which light can no longer escape the gravitational field. Any physical object whose radius $R$ becomes less than or equal to the Schwarzschild radius will undergo gravitational collapse and become a black hole.

**Flamm's paraboloid**

The spatial curvature of the Schwarzschild solution for $r > r_s$ can be visualized as the graphic shows. Consider a constant time equatorial slice through the Schwarzschild solution ($\theta = \pi/2, \ t = \text{constant}$) and let the position of a particle moving in this plane be described with the remaining Schwarzschild coordinates $(r, \ \phi)$. Imagine now that there is an additional Euclidean dimension $w$, which has no physical reality (it is not part of spacetime). Then replace the $(r, \ \phi)$ plane with a surface dimpled in the $w$ direction according to the equation (Flamm's paraboloid).
This surface has the property that distances measured within it match distances in the Schwarzschild metric, because with the definition of $w$ above,

$$dw^2 + dr^2 + r^2 d\phi^2 = -c^2 dt^2 = \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\phi^2$$

Thus, Flamm's paraboloid is useful for visualizing the spatial curvature of the Schwarzschild metric. It should not, however, be confused with a gravity well. No ordinary (massive or massless) particle can have a worldline lying on the paraboloid, since all distances on it are spacelike (this is a cross-section at one moment of time, so all particles moving across it must have infinite velocity). Even a tachyon would not move along the path that one might naively expect from a "rubber sheet" analogy: in particular, if the dimple is drawn pointing upward rather than downward, the tachyon's path still curves toward the central mass, not away. See the gravity well article for more information.

Flamm's paraboloid may be derived as follows. The Euclidean metric in the cylindrical coordinates $(r, \varphi, w)$ is written

$$ds^2 = dw^2 + dr^2 + r^2 d\phi^2.$$  

Letting the surface be described by the function $w = w(r)$, the Euclidean metric can be written as

$$ds^2 = \left[1 + \left(\frac{dw}{dr}\right)^2\right] dr^2 + r^2 d\phi^2,$$

Comparing this with the Schwarzschild metric in the equatorial plane ($\theta = \pi/2$) at a fixed time ($t =$ constant, $dt = 0$)

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\phi^2,$$

yields an integral expression for $w(r)$:

$$w(r) = \int \frac{dr}{\sqrt{\frac{r}{r_s} - 1}} = 2r_s \sqrt{\frac{r}{r_s} - 1 + \text{constant}}$$

whose solution is Flamm's paraboloid.

**Orbital motion**

*For more details on this topic, see Schwarzschild geodesics.*

A particle orbiting in the Schwarzschild metric can have a stable circular orbit with $r > 3r_s$. Circular orbits with $r$ between $3r_s / 2$ and $3r_s$ are unstable, and no circular orbits exist for $r < 3r_s / 2$. The circular orbit of minimum radius $3r_s / 2$ corresponds to an orbital velocity approaching the speed of
light. It is possible for a particle to have a constant value of \( r \) between \( r_s \) and \( 3r_s / 2 \), but only if some force acts to keep it there.

Noncircular orbits, such as Mercury's, dwell longer at small radii than would be expected classically. This can be seen as a less extreme version of the more dramatic case in which a particle passes through the event horizon and dwells inside it forever. Intermediate between the case of Mercury and the case of an object falling past the event horizon, there are exotic possibilities such as "knife-edge" orbits, in which the satellite can be made to execute an arbitrarily large number of nearly circular orbits, after which it flies back outward.

**Symmetries**

The group of isometries of the Schwarzschild metric is the subgroup of the ten-dimensional Poincaré group which takes the time axis (trajectory of the star) to itself. It omits the spatial translations (three dimensions) and boosts (three dimensions). It retains the time translations (one dimension) and rotations (three dimensions). Thus it has four dimensions. Like the Poincaré group, it has four connected components: the component of the identity; the time reversed component; the spatial inversion component; and the component which is both time reversed and spatially inverted.

**Quotes**

"Es ist immer angenehm, über strenge Lösungen einfacher Form zu verfügen." (It is always pleasant to have exact solutions in simple form at your disposal.) – Karl Schwarzschild, 1916.

**See also**

- Deriving the Schwarzschild solution
- Reissner–Nordström metric (charged, non-rotating solution)
- Kerr metric (uncharged, rotating solution)
- Kerr–Newman metric (charged, rotating solution)
- BKL singularity (interior solution)
- Black hole, a general review
- Schwarzschild coordinates
- Kruskal–Szekeres coordinates
- Eddington–Finkelstein coordinates
- Gullstrand–Painlevé coordinates
- Lemaître coordinates (Schwarzschild solution in synchronous coordinates)
- Frame fields in general relativity (Lemaître observers in the Schwarzschild vacuum)

**Notes**

References

  - scan of the original paper
  - text of the original paper, in Wikisource
  - translation by Antoci and Loinger
  - a commentary on the paper, giving a simpler derivation


- Flamm, L (1916). "Beiträge zur Einstein'schen Gravitationstheorie". *Physikalische Zeitschrift* 17: 448–?.


Categories: Black holes | Exact solutions in general relativity

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