Riemannian geometry
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Elliptic geometry is also sometimes called "Riemannian geometry".

Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds, smooth manifolds with a Riemannian metric, i.e. with an inner product on the tangent space at each point which varies smoothly from point to point. This gives, in particular, local notions of angle, length of curves, surface area, and volume. From those some other global quantities can be derived by integrating local contributions.

Riemannian geometry originated with the vision of Bernhard Riemann expressed in his inaugurational lecture *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* (English: On the hypotheses on which geometry is based). It is a very broad and abstract generalization of the differential geometry of surfaces in \( \mathbb{R}^3 \). Development of Riemannian geometry resulted in synthesis of diverse results concerning the geometry of surfaces and the behavior of geodesics on them, with techniques that can be applied to the study of differentiable manifolds of higher dimensions. It enabled Einstein's general relativity theory, made profound impact on group theory and representation theory, as well as analysis, and spurred the development of algebraic and differential topology.

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### Introduction
Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century. It deals with a broad range of geometries whose metric properties vary from point to point, as well as two standard types of Non-Euclidean geometry, spherical geometry and hyperbolic geometry, as well as Euclidean geometry itself.

Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the more complicated structure of pseudo-Riemannian manifolds, which (in four dimensions) are the main objects of the theory of general relativity. Other generalizations of Riemannian geometry include Finsler geometry and spray spaces.

The following articles provide some useful introductory material:

1. Metric tensor
2. Riemannian manifold
3. Levi-Civita connection
4. Curvature
5. Curvature tensor
6. List of differential geometry topics
7. Glossary of Riemannian and metric geometry.

**Classical theorems in Riemannian geometry**

What follows is an incomplete list of the most classical theorems in Riemannian geometry. The choice is made depending on its importance, beauty, and simplicity of formulation. Most of the results can be found in the classic monograph by Jeff Cheeger and D. Ebin (see below).

The formulations given are far from being very exact or the most general. This list is oriented to those who already know the basic definitions and want to know what these definitions are about.

**General theorems**

1. **Gauss–Bonnet theorem** The integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold is equal to $2\pi \chi(M)$ where $\chi(M)$ denotes the Euler characteristic of $M$. This theorem has a generalization to any compact even-dimensional Riemannian manifold, see generalized Gauss-Bonnet theorem.
2. **Nash embedding theorems** also called fundamental theorems of Riemannian geometry. They state that every Riemannian manifold can be isometrically embedded in a Euclidean space $\mathbb{R}^n$.

**Geometry in large**

In all of the following theorems we assume some local behavior of the space (usually formulated using curvature assumption) to derive some information about the global structure of the space, including either some information on the topological type of the manifold or on the behavior of points at "sufficiently large" distances.

**Pinched sectional curvature**
1. **Sphere theorem.** If $M$ is a simply connected compact $n$-dimensional Riemannian manifold with sectional curvature strictly pinched between $1/4$ and $1$ then $M$ is diffeomorphic to a sphere.

2. **Cheeger's finiteness theorem.** Given constants $C, D$ and $V$, there are only finitely many (up to diffeomorphism) compact $n$-dimensional Riemannian manifolds with sectional curvature $\left| K \right| \leq C$, diameter $\leq D$ and volume $\geq V$.

3. **Gromov's almost flat manifolds.** There is an $\varepsilon_n > 0$ such that if an $n$-dimensional Riemannian manifold has a metric with sectional curvature $\left| K \right| \leq \varepsilon_n$ and diameter $\leq 1$ then its finite cover is diffeomorphic to a nil manifold.

### Sectional curvature bounded below

1. **Cheeger-Gromoll's Soul theorem.** If $M$ is a non-compact complete non-negatively curved $n$-dimensional Riemannian manifold, then $M$ contains a compact, totally geodesic submanifold $S$ such that $M$ is diffeomorphic to the normal bundle of $S$ (S is called the soul of $M$.) In particular, if $M$ has strictly positive curvature everywhere, then it is diffeomorphic to $\mathbb{R}^n$. G. Perelman in 1994 gave an astonishingly elegant/short proof of the Soul Conjecture: $M$ is diffeomorphic $\mathbb{R}^n$ if it has positive curvature at only one point.

2. **Gromov's Betti number theorem.** There is a constant $C = C(n)$ such that if $M$ is a compact connected $n$-dimensional Riemannian manifold with positive sectional curvature then the sum of its Betti numbers is at most $C$.

3. **Grove–Petersen's finiteness theorem.** Given constants $C, D$ and $V$, there are only finitely many homotopy types of compact $n$-dimensional Riemannian manifolds with sectional curvature $K \geq C$, diameter $\leq D$ and volume $\geq V$.

### Sectional curvature bounded above

1. The **Cartan–Hadamard theorem** states that a complete simply connected Riemannian manifold $M$ with nonpositive sectional curvature is diffeomorphic to the Euclidean space $\mathbb{R}^n$ with $n = \text{dim} M$ via the exponential map at any point. It implies that any two points of a simply connected complete Riemannian manifold with nonpositive sectional curvature are joined by a unique geodesic.

2. The geodesic flow of any compact Riemannian manifold with negative sectional curvature is ergodic.

3. If $M$ is a complete Riemannian manifold with sectional curvature bounded above by a strictly negative constant $k$ then it is a CAT($k$) space. Consequently, its fundamental group $\Gamma = \pi_1(M)$ is Gromov hyperbolic. This has many implications for the structure of the fundamental group:

   - it is finitely presented;
   - the word problem for $\Gamma$ has a positive solution;
   - the group $\Gamma$ has finite virtual cohomological dimension;
   - it contains only finitely many conjugacy classes of elements of finite order;
   - the abelian subgroups of $\Gamma$ are virtually cyclic, so that it does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

### Ricci curvature bounded below

1. **Myers theorem.** If a compact Riemannian manifold has positive Ricci curvature then its fundamental group is finite.

2. **Splitting theorem.** If a complete $n$-dimensional Riemannian manifold has nonnegative Ricci
curvature and a straight line (i.e. a geodesic which minimizes distance on each interval) then it is isometric to a direct product of the real line and a complete \((n-1)\)-dimensional Riemannian manifold which has nonnegative Ricci curvature.

3. **Bishop–Gromov inequality.** The volume of a metric ball of radius \(r\) in a complete \(n\)-dimensional Riemannian manifold with positive Ricci curvature has volume at most that of the volume of a ball of the same radius \(r\) in Euclidean space.

4. **Gromov's compactness theorem.** The set of all Riemannian manifolds with positive Ricci curvature and diameter at most \(D\) is pre-compact in the Gromov-Hausdorff metric.

### Negative Ricci curvature

1. The isometry group of a compact Riemannian manifold with negative Ricci curvature is discrete.
2. Any smooth manifold of dimension \(n \geq 3\) admits a Riemannian metric with negative Ricci curvature.[1] *(This is not true for surfaces.)*

### Positive scalar curvature

1. The \(n\)-dimensional torus does not admit a metric with positive scalar curvature.
2. If the injectivity radius of a compact \(n\)-dimensional Riemannian manifold is \(\geq \pi\) then the average scalar curvature is at most \(n(n-1)\).

### See also

- Shape of the universe
- Basic introduction to the mathematics of curved spacetime
- Normal coordinates
- Systolic geometry
- Riemann-Cartan geometry in Einstein-Cartan theory (Motivation)

### Notes

1. ^ Joachim Lohkamp has shown (Annals of Mathematics, 1994) that any manifold of dimension greater than two admits a metric of negative Ricci curvature.

### References

**Books**


- Cheeger, Jeff; Ebin, David G. (2008), *Comparison theorems in Riemannian geometry*, Providence, RI: AMS Chelsea Publishing; Revised reprint of the 1975 original.


**Papers**


**External links**

- Weisstein, Eric W., "Riemannian Geometry" from MathWorld.


Categories: Riemannian geometry